TECHNICAL NOTE

A calculation procedure for momentum and heat transfer in the turbulent boundary layer of gases using the methods of lines and control volumes (MOLCV)

Carlos Schuler
Departamento de Termodinamica, Universidad Simon Bolivar, Caracas, Venezuela

Antonio Campo
Department of Mechanical Engineering, Florida International University, Miami, FL 33199, USA

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Introduction

The prediction of turbulent boundary layers involving heat transfer has many applications in engineering practice. The art of predicting the hydrodynamic and thermal behavior of turbulent boundary layers has advanced very rapidly with the advent of large-scale digital computers and the development of sophisticated finite-difference procedures (explicit and implicit) for the solution of boundary layer equations.[1]

The general objective of this investigation was to implement a hybrid computational procedure of explicit type for the rapid solution of the turbulent boundary layer equations. A procedure involving the method of lines[2] has been utilized and appropriately modified for the treatment of the conservation equations of mass, momentum, and energy. A variant of this procedure is introduced wherein the discretization process of the participating transversal derivatives in the conservation equations was carried out via control volumes.[3] Accordingly, the control volumes have to be constructed in such a way that their sizes have infinitesimal length and finite height. Then the conservation equations are rewritten as a system of first-order ordinary differential equations, where the streamwise coordinate is the independent variable with a continuous behavior. In turn, the resulting initial-value problem may be readily integrated numerically with a standard Runge–Kutta algorithm on a personal computer. In passing, it should be mentioned that the calculation procedure presented in Ref. 4 resembles the one presented here. However, their discretization is more involved, necessitating the approximation of both differential and integral terms in the conservation equations.

Ultimately, the experimental data most heavily relied upon for comparison purposes were expressed in terms of global quantities, such as the local skin friction coefficient and the Stanton number for an air flow along a flat plate covering a range of Reynolds numbers from 10^5 to 10^7.

Governing equations

Under the assumption of constant properties, the time-average conservation equations of mass, momentum, and energy for a turbulent boundary layer along a flat plate may be written in a general form as follows:

$$\frac{\partial}{\partial x}(u\phi)+\frac{\partial}{\partial y}(v\phi)=\frac{\partial}{\partial y}\left(\Gamma_{\text{eff}}\frac{\partial\phi}{\partial y}\right)+S_{\phi} \tag{1}$$

The isothermal flat plate maintained at t_w is aligned parallel to an air stream having uniform velocity u_∞ and uniform temperature t_∞.

The simplest model available for the turbulent momentum exchange process, i.e., Prandtl's mixing length concept, has been adopted in this paper:

$$\varepsilon_M = l^2\left|\frac{\partial u}{\partial y}\right| \tag{2}$$

In order to account for the viscous sublayer of the inner region in the boundary layer, a modified expression for the mixing length[5] has been used here; i.e.,

$$l_i = 0.4y[1-\exp(-y^+/A^+)] \tag{3a}$$

where $A^+=25$. Alternatively, the outer region of the boundary layer has been modeled by a constant mixing length[6] given by

$$l_0 = 0.086\delta \tag{3b}$$

Additionally, the turbulent Prandtl number has been taken as 0.9.

Computational methodology

It is important to recognize that, although turbulence models may be carefully formulated, sometimes predictions cannot be improved beyond a certain point because of the inherent limitations of the numerical methodology employed. Consequently, the merits of a higher-order closure turbulence model can be easily lost in the discretization process.

The new proposed method MOLCV

Briefly, the method of lines is essentially a hybrid technique for replacing a partial differential equation by a system of ordinary differential equations in one of the independent variables.[2] If the equation is parabolic—having independent variables x (axial variable) and y (transversal variable), as in Equation 1—the partial derivatives with respect to y may be replaced by finite-difference expressions. Consequently, this systematic pro-

Address reprint requests to Professor Campo at the Department of Mechanical Engineering, Florida International University, Miami, FL 33199, USA.

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cedure automatically gives rise to a system of first-order ordinary differential equations, where the independent variable is x.

Alternatively, the discretization procedure of the conservation equations (Equation 1) may be carried out more efficiently using the control volume approach.[3] This combination gives rise to the method of lines and control volumes (MOLCV) wherein the control volume has infinitesimal length in the axial direction and finite height in the transversal direction. Correspondingly, the general transport equation (Equation 1) integrated between the appropriate limits of s and n, leads to

$$\int_s^n \frac{\partial}{\partial x}(u\phi)dy + (v\phi)_n - (v\phi)_s = \left(\Gamma_{eff}\frac{\partial\phi}{\partial y}\right)_n - \left(\Gamma_{eff}\frac{\partial\phi}{\partial y}\right)_s + \int_s^n S_\phi dy \qquad (4)$$

rearranging terms yields the ordinary differential equation

$$\frac{d}{dx}(u_P\phi_P) = \frac{1}{\Delta y}\left(\Gamma_{eff}\frac{\partial\phi}{\partial y} - (v\phi)\right)_s^n + S_{\phi P} \qquad (5)$$

Thus, from a strict mathematical point of view, Equation 5 governs the continuous variation of each transported quantity in the x-direction at any fixed distance $y = y_P$ from the plate.

In addition, the diffusive term appearing at the upper and lower faces of the control volume are described by an appropriate logarithmic law for the variation of each transported quantity between neighboring lines.

Transformation of the basic equations

Equation 5, rewritten for each of the conservation equations, results in the following set of ordinary first-order differential equations.

Momentum:
$$\frac{du_P}{dx} = \frac{1}{\Delta y(2u_P - u_n)}\left\{\left[(v+\varepsilon_M)\frac{\partial u}{\partial y}\right]_s^n - v_s(u_s - u_n)\right\} \qquad (6)$$

Mass:
$$v_n = v_s - \Delta y\frac{du_P}{dx} \qquad (7)$$

Energy:
$$\frac{dt_P}{dx} = \frac{1}{u_P\Delta y}\left[\left(\frac{1}{Pr} + \frac{1}{Pr_t}\frac{\varepsilon_M}{v}\right)\frac{\partial t}{\partial y} - vt\right]_s^n - \frac{t_P}{u_P}\frac{du_P}{dx} \qquad (8)$$

Examination of Equations 6–8 reveals that the transversal velocity v and the axial velocity gradient du_P/dx may be explicitly calculated.

In addition, the velocity field in the vicinity of the wall has been examined, assuming the existence of a Couette flow. According to Ref. 7, this provides the near-wall boundary conditions for the turbulent velocity field based on experimental observations.

Thus, the partially discretized momentum and energy transport equations constitute a coupled system of first-order ordinary differential equations. Correspondingly, these equations can be numerically integrated explicitly with a Runge–Kutta fourth-order algorithm on a personal computer.

Finally, the necessary number of lines inside the boundary layer may be determined for each axial step by adding new lines to a preselected number of lines at the leading edge, such that the condition at the outer edge $u/u_\infty > 0.99$ is always satisfied. Thus the increasing number of lines serve to delineate step-by-step the natural growth of the turbulent boundary layers.

Calculated and experimental results

The numerical solutions using MOLCV were performed for $10^5 < Re_x < 10^7$ and $Pr = 0.7$. Undoubtedly, the most demanding test of the marching procedure (Runge–Kutta algorithm) occurs in the neighborhood of the trailing edge of the plate.

We now turn to the experimentally and numerically deter-

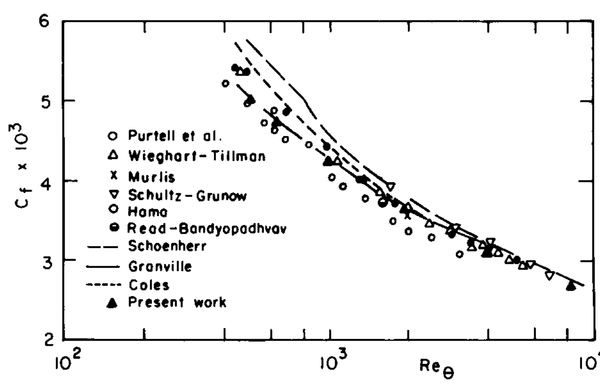

Figure 1 Local skin-friction coefficient

Notation

- A^+ Van Driest constant
- C_f Local skin-friction coefficient
- l Mixing length
- Pr Prandtl number
- Pr_t Turbulent Prandtl number
- Re_x Local Reynolds number
- Re_θ Momentum thickness Reynolds number
- S_ϕ Source term (Equation 1)
- t Temperature
- t_w Wall temperature
- t_∞ Free stream temperature
- u Axial velocity component
- u_∞ Free stream velocity
- v Transversal velocity component
- x Axial coordinate
- y Transversal coordinate
- y^+ Nondimensional transversal coordinate

Greek letters

- Γ_{eff} Effective diffusion coefficient (Equation 1)
- Δy Transversal interval
- δ Thickness of the momentum boundary layer
- ε_M Eddy diffusivity for momentum
- v Kinematic viscosity
- ρ Density
- ϕ Generalized variable (Equation 1)

Subscripts

- i Inner
- n North face
- o Outer
- P Line
- s South face

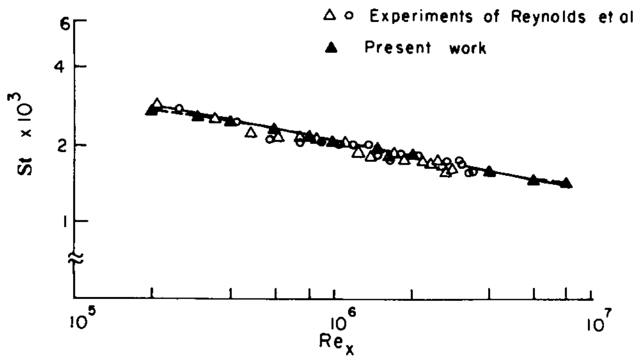

Figure 2 Stanton number

mined values of global quantities, which are brought together in Figures 1 and 2. Figure 1 compares the experimental and numerical results for the variation of skin friction coefficient, C_f, with the momentum thickness Reynolds number, Re_θ. The vast number of data points for air collected and presented in Ref. 8 is reproduced here. An overall inspection of Figure 1 reveals that excellent agreement prevails between both sets of results. Additionally, in Figure 2, heat transfer results for air in terms of the Stanton number, St, are plotted as a function of the local Reynolds number, Re_x. Figure 2 confirms the expected monotonic decrease of the turbulent heat transfer coefficient along the plate. Note that there is no appreciable deviation between the numerical hybrid solution and the experiments reported in Ref. 9.

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